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Short Communication

Preliminary analytical and numerical investigations of a van der Pol type oscillator having discontinuous dependence on the velocity

R.E. Mickens[a,*], K. Oyedeji[b], S.A. Rucker[c]

[a]*Department of Physics, Clark Atlanta University, P.O. Box 172, Atlanta, GA 30314, USA*
[b]*Department of Physics, Morehouse College, Atlanta, GA 30314, USA*
[c]*Department of Mathematical Sciences, Clark Atlanta University, Atlanta, GA 30314, USA*

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The van der Pol equation [1,2]

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \tag{1}$$

where ε is a positive parameter, provides a model of a one-dimensional oscillatory system having a unique limit cycle. It is of interest, both mathematically and from the viewpoint of future applications to the natural and engineering sciences, to consider generalizations of this equation. A nontrivial extension of Eq. (1) is

$$\ddot{x} + x = \varepsilon(1 - x^2)\,\mathrm{sign}(\dot{x}), \tag{2}$$

where the "sign function" is defined to be, for real z,

$$\mathrm{sign}(z) = \begin{cases} +1, & z > 0, \\ 0, & z = 0, \\ -1, & z < 0. \end{cases} \tag{3}$$

The purpose of this communication is to present the results of our preliminary investigations on Eq. (2). In particular, several of the mathematical properties related to this equation's solutions

*Corresponding author. Tel.: +1-440-880-6923; fax: +1-404-880-6258.
E-mail address: rohrs@math.gatech.edu (R.E. Mickens).

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are obtained, an approximate analytical solution is calculated using the method of first-order averaging, and the equation is numerically integrated by use of the nonstandard method of Mickens [3].

The coupled, first-order system equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + \varepsilon(1 - x^2)\,\mathrm{sign}\,(y), \tag{4}$$

and the equation determining the paths of trajectories in the (x, y) phase space is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x + \varepsilon(1 - x^2)\,\mathrm{sign}\,(y)}{y}. \tag{5}$$

Observe that Eq. (5) is invariant under the transformation

$$x \to -x, \quad y \to -y. \tag{6}$$

This transformation corresponds to inversion through the origin and implies that if $(x(t), y(t))$ is a possible trajectory in phase-space, then $(-x(t), -y(t))$ is also a trajectory [4].

From Eq. (4) it follows that Eq. (2) has a fixed point, or equilibrium solution, at $(\bar{x}, \bar{y}) = (0, 0)$. The stability of this fixed point can be determined by use of the following "energy" argument [4]. Define $R(x, y)$ as

$$R(x, y) = \left(\frac{1}{2}\right)(x^2 + y^2). \tag{7}$$

Differentiating this expression and replacing the derivatives, $\dot{x}$ and $\dot{y}$, by the results in Eq. (4), gives

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \varepsilon|y|(1 - x^2). \tag{8}$$

Note that for $|x| \ll 1$, $\mathrm{d}R/\mathrm{d}t \geqslant 0$, while for $|x| > 1$, then $\mathrm{d}R/\mathrm{d}t < 0$. The first fact implies that the fixed point at $(\bar{x}, \bar{y}) = (0, 0)$ is unstable. The second result indicates that trajectories far from the origin in phase space are attracted back to a neighborhood of the fixed space. This analysis [2,4] indicates that Eq. (2) may have a limit-cycle solution.

An analytic approximation to the solutions of Eq. (2) can be determined by use of the method of first-order averaging, also known as the method of slowly varying amplitude and phase [1,2,4]. The assumed solution takes the form

$$x(t, \varepsilon) = a(t, \varepsilon)\cos[t + \phi(t, \varepsilon)], \tag{9}$$

where ε is taken to be small, i.e.,

$$0 < \varepsilon \ll 1. \tag{10}$$

The first-order expressions for $a(t, \varepsilon)$ and $\phi(t, \varepsilon)$ are [2]

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\left(\frac{\varepsilon}{2\pi}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\sin\psi\,\mathrm{d}\psi, \tag{11a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\left(\frac{\varepsilon}{2\pi a}\right)\int_0^{2\pi} F(a\cos\psi, -a\sin\psi)\cos\psi\,\mathrm{d}\psi, \tag{11b}$$

where, for Eq. (2), F is

$$\begin{aligned} F(x,\dot{x}) \to F(a\cos\psi, -a\sin\psi) \\ &= (1-a^2\cos^2\psi)\,\mathrm{sign}\,(-a\sin\psi) \\ &= -(1-a^2\cos^2\psi)\,\mathrm{sign}\,(a\sin\psi). \end{aligned} \tag{12}$$

A direct, but elementary calculation gives

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{3\varepsilon}{4\pi}\right)[\mathrm{sign}\,(a)]\left[\tfrac{8}{3} - a^2\right], \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = 0. \tag{13}$$

The solution for $\phi(t,\varepsilon)$ is

$$\phi(t,\varepsilon) = \phi_0 = \text{constant}. \tag{14}$$

The first equation in Eq. (13) can be solved using the method of variable separation. Carrying out this calculation gives

$$a(t) = \sqrt{\frac{8}{3}}\left[\frac{\left(a_0 + \sqrt{\frac{8}{3}}\right) + \left(a_0 - \sqrt{\frac{8}{3}}\right)\exp\left(-\sqrt{6}\varepsilon t/\pi\right)}{\left(a_0 + \sqrt{\frac{8}{3}}\right) - \left(a_0 - \sqrt{\frac{8}{3}}\right)\exp\left(-\sqrt{6}\varepsilon t/\pi\right)}\right]. \tag{15}$$

Note that the asymptotic value of the amplitude is

$$\lim_{t\to\infty} a(t) \longrightarrow \sqrt{\tfrac{8}{3}} = 1.633. \tag{16}$$

These results clearly indicate the existence of a limit cycle for Eq. (2). After initial transients damp out, all the solutions approach the (approximate) periodic solution

$$x(t,\varepsilon) \simeq \sqrt{\tfrac{8}{3}}\cos(t+\phi_0). \tag{17}$$

In contrast, the standard van der Pol equation has the first-order in ε asymptotic solution [1,2,4]

$$x_{\mathrm{vaP}}(t,\varepsilon) \simeq 2\cos(t+\phi_0). \tag{18}$$

The above solution for Eq. (2) indicates that a stable limit cycle exists.

The following numerical integration scheme was used to numerically integrate Eqs. (4):

$$x_{k+1} = \psi x_k + \phi y_k, \tag{19a}$$

$$y_{k+1} = \psi y_k - x_{k+1} + \varepsilon[1-(x_{k+1})^2]\,\mathrm{sign}\,(y_k), \tag{19b}$$

where Δt = step size in time; x_k and y_k are, respectively, approximations to $x(t_k)$ and $y(t_k)$, where $t_k = (\Delta t)k$, with k-integer valued; and the functions ψ and ϕ are

$$\psi = \cos(\Delta t), \quad \phi = \sin(\Delta t). \tag{20}$$

This numerical procedure is based on the nonstandard methods of Mickens [3]. Figs. 1 and 2 present typical numerical solutions. For Fig. 1, the initial conditions are $(x_0, y_0) = (0.0, 0.1)$ and this situation gives a trajectory in the (x, y) phase-plane spiraling out to approach the

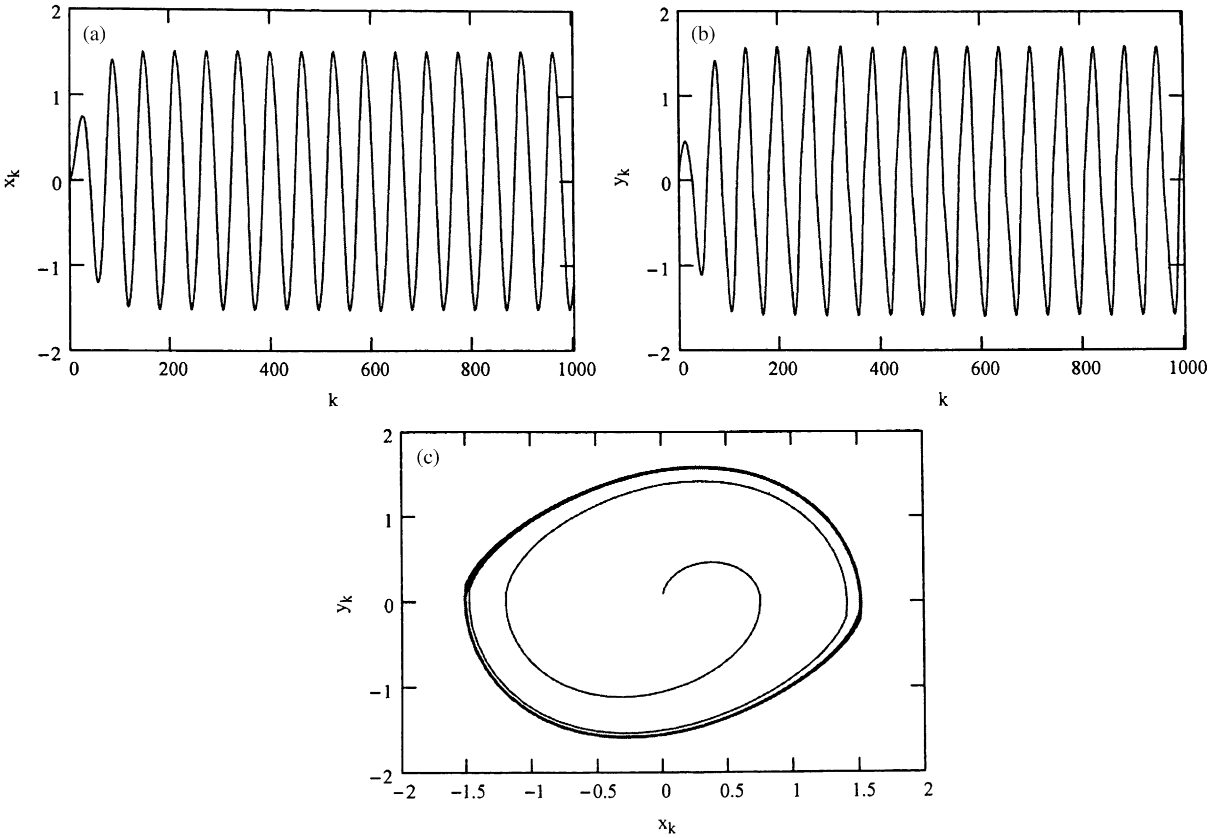

Fig. 1. (a) Plot of x_k versus k. (b) Plot of y_k versus k. (c) Phase-space plot of y_k versus x_k. The values of the parameters are $\varepsilon = 0.5$, $\Delta t = 0.01$ with initial conditions $x_0 = 0.0$ and $y_0 = 0.1$.

limit cycle. Similarly, in Fig. 2, the initial conditions $(x_0, y_0) = (0.0, 2.0)$ correspond to trajectories approaching the limit cycle from the outside. All of these results are consistent with the analytical approximation derived from application of the method of first-order averaging.

In summary, both the method of first-order averaging and the numerical integration results indicate that Eq. (2) has a unique, stable limit cycle determined from the numerical integration is approximately 1.52, while the value from Eq. (16) is 1.63. This is excellent agreement if note is made of the fact that the nonlinear term in Eq. (2) is a discontinuous function of the velocity, $\dot{x}$. A future research topic will be to see if higher-order averaging techniques can be applied to Eq. (2). This may provide better agreement between the numerical and analytical calculated values for the limit-cycle amplitude under the requirement $0 < \varepsilon \ll 1$. Since Eq. (2) is not of a form to which the Liénard–Levinson–Smith theorem [2] can be applied, it would be of interest to derive a theorem, along with its proof, applicable for this situation.

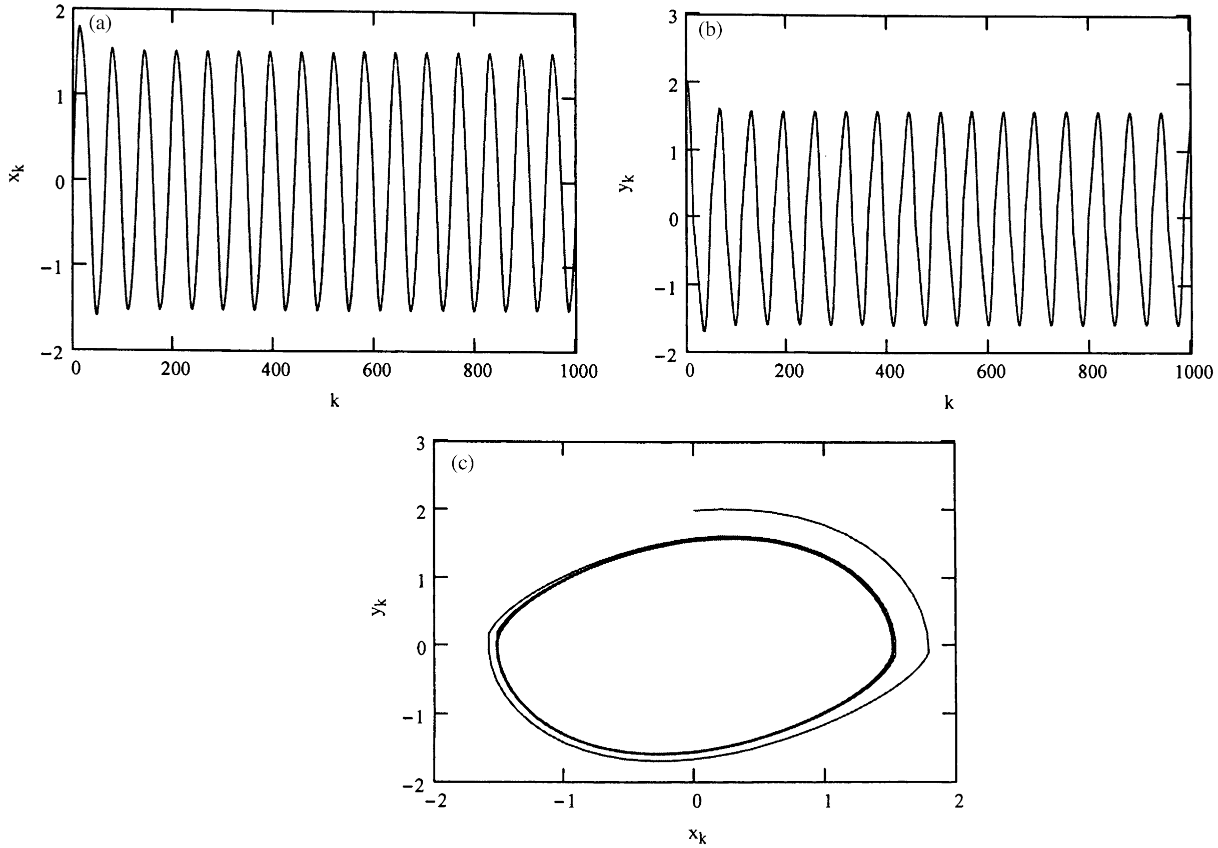

Fig. 2. (a) Plot of x_k versus k. (b) Plot of y_k versus k. (c) Phase-space plot of y_k versus x_k. The values of the parameters are $\varepsilon = 0.5$, $\Delta t = 0.1$, with initial conditions $x_0 = 0$ and $y_0 = 2.0$.

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